Optimal Polygonal Representation of Planar Graphs

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Abstract. In this paper, we consider the problem of representing graphs by polygons whose sides touch. We show that at least six sides per polygon are necessary by constructing a class of planar graphs which cannot be represented by pentagons. We next show that the lower bound of six sides is matched by an upper bound of six sides with a linear time algorithm for representing any planar graph by touching hexagons. Moreover, our algorithm produces convex polygons with edges having the slopes of a regular hexagon, and with an area bound of $n \times n$ for fully triangulated graphs.

1 Introduction

For both theoretical and practical reasons, there is a large body of work considering how to represent planar graphs as *contact graphs*, i.e., graphs whose vertices are represented by geometrical objects with edges corresponding to two objects touching in some specified fashion. Typical classes of objects might be curves, line segments or isothetic rectangles and an early result is Koebe's theorem [15], which shows that all planar graphs can be represented by touching disks.

In this paper, we consider contact graphs whose objects are simple polygons, with an edge occurring whenever two polygons have non-trivially overlapping sides. As with treemaps [3], such representations are preferred in some contexts [4] over the standard node-link representations for displaying relational information. Using adjacency to represent connection can be much more compelling, and cleaner, than drawing a line segment between two nodes. For ordinary users, this representation suggests the familiar metaphor of a geographical map.

It is obvious that any graph represented this way must be planar. As noted by de Fraysseix *et al.* [6], it is also easy to see that all planar graphs have such representations for sufficiently general polygons. Starting with a straight-line planar drawing of a graph, we can create a polygon for each vertex by taking the midpoints of all adjacent edges and the centers of all neighboring faces. Note that the number of sides in each such polygon is proportional to the degree of its vertex. Moreover, these polygons are not necessarily convex; see Figure 1.

It is desirable, for aesthetic, practical and cognitive reasons, to limit how complicated the polygons are. Fewer sides, as well as wider angles in the polygons, make for simpler and cleaner drawings. In related applications such as floor-planning [19], physical constraints make polygons with very small angles or many sides undesirable. One is then led to consider how simple such representations can be. How many sides do we really need? Can we insist that the polygons be convex, perhaps with a lower bound on the size of the angles or the edges? If limiting some of these parameters prevents the drawings of all planar graphs, which ones can be drawn?

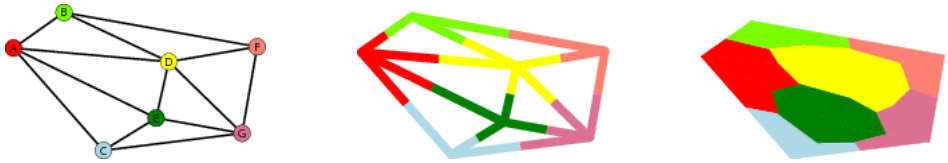

Fig. 1. Given a drawing of a planar graph(a), we apportion the edges to the endpoints by cutting each edge in half (b), and then apportion the faces to form polygons (c).

This paper provides a partial answer to these questions. Previously, it was known [9, 19] that all planar graphs can be represented using non-convex octagons. On the other hand, it is not hard to see that one cannot use triangles (e.g., K_5 minus one edge cannot be represented with triangles). Our main result is showing that hexagons are necessary and sufficient for representing all planar graphs. For sufficiency, we describe a linear time algorithm that produces a representation using convex hexagon all of whose sides have slopes 1, 0 or -1. We show that the area requirement is $n \times n$ for fully triangulated graphs. This algorithm is based on an algorithm of Kant [13] for embedding graphs on a hexagonal grid.

1.1 Related Work

As remarked above, there is a rich literature related to various types of contact graphs. There are many results considering curves and line segments as objects (cf. [10, 11]). For closed shapes such as polygons, results are rarer, except for axis-aligned (or *isothetic*) rectangles. In a sense, results on representing planar graphs as "contact systems" can be dated back to Koebe's 1936 theorem [15] which states that any planar graph can be represented as a contact graph of disks in the plane.

The focus of this paper is on side-to-side contact of polygons. The algorithms of He [9] and Liao *et al.* [19] produce contact graphs of this type for any planar graph, with nodes represented by the union of at most two isothetic rectangles, thus giving a polygonal representation by non-convex octagons. By relaxing the isothetic constraint to allow angles of 45° we are able to reduce the number of sides to six, while enforcing convexity.

Although not considered by the authors, an upper bound of six for the minimum number of sides in a touching polygon representation of planar graphs might be obtained from the vertex-to-side triangle contact graphs of de Fraysseix *et al.* [6]. The top edge of each triangle can be converted into a raised 3-segment polyline, clipping the tips of the triangles touching it from above, thereby turning the triangles into side-touching hexagons. It is likely to be difficult to use this approach for generating hexagonal representations as it involves computing the amounts by which each triangle may be raised so as to become a hexagon without changing any of the adjacencies. Moreover, by the nature of such an algorithm, there would be many "holes," potentially making such drawings less appealing, or requiring further modifications to remove them.

We now turn to contact graphs using isothetic rectangles, which are often referred to as *rectangular layouts*. This is the most extensively studied class of contact graphs, due, in part, to the relation to application areas such as VLSI floor-planning [17, 25], architectural design [23] and geographic information systems [7], but also due the mathematical ramifications and connections to other areas such as rectangle of influence drawings [20] and proximity drawings [1, 12].

Graphs allowing rectangular layouts have been fully characterized [21, 24] with linear algorithms for deciding if a rectangular layout is possible and, if so, constructing one. The simplest formulation [4] notes that a graph has a rectangular layout if and only if it has a planar embedding with no filled triangles. Thus, K_4 has no rectangular layout. Buchsbaum *et al.* [4] also show, using results of Biedl *et al.* [2], that graphs that admit rectangular layouts are precisely those that admit a weaker variation of planar rectangle of influence drawings.

Rectangular layouts required to form a partition of a rectangle are known as *rectangular duals*. In a sense, these are "maximal" rectangular layouts; many of the results concerning rectangular layouts are built on results concerning rectangular duals. Graphs admitting rectangular duals have been characterized [8, 16, 18] and there are linear time algorithms [8, 14] for constructing them.

Another view of rectangular layouts arises in VLSI floor-planning, where a rectangle is partitioned into rectilinear regions so that region adjacencies correspond to a given planar graph. It is natural to try to minimize the number of sides of the resulting regions. The best known results are due to He [9] and Liao *et al.* [19] who show that regions need not have more than 8 sides. Both of these algorithms run in $O(n)$ time and produce layouts on an integer grid of size $O(n) \times O(n)$, where n is the number of vertices.

Rectilinear cartograms can be defined as rectilinear contact graphs for vertex-weighted planar graphs, where the area of a rectilinear region must be proportional to the weight of its corresponding node. Even with this extra condition, de Berg *et al.* [5] show that rectilinear cartograms can always be constructed in $O(n \log n)$ time, using regions having at most 40 sides.

2 Lower Bound of Six Sides

Here we show that at least six sides per polygon are needed in touching polygon representations of planar graphs. We begin by constructing a class of planar graphs that cannot be represented by four-sided polygons and then extend the argument to show that there exists a class of planar graphs that cannot be represented by five-sided regions.

2.1 Four Sides Are Not Enough

Consider the fully triangulated graph G in Figure 2(a). G has three nodes on the outer face A, B and C, and contains a chain of nodes $1, ..., k$ which are all adjacent to A and B. Consecutive nodes in the chain, i and $i+1$, are also adjacent. The remaining nodes of G are degree-3 nodes l_i and r_i inside the triangles $\Delta(A, i, i+1)$ and $\Delta(B, i, i+1)$.

Theorem 1. *For k sufficiently large, there does not exist a touching polygon representation for G in which all regions have complexity 4 or less.*

Proof: Assume, for the sake of contradiction, that we are given a touching polygon drawing for G in which all regions have complexity 4 or less. Without loss of generality, we assume that the drawing has an embedding that corresponds to the one shown in Figure 2(a). Let Q_A and Q_B denote the quadrilaterals representing nodes A and B, and Q_i denotes the quadrilateral representing node i. Once again, without loss of generality, let Q_A lie in the left corner, Q_B in the right corner and Q_C at the top of the drawing.

We start with a couple of observations:

Observation 1: Since the three quadrilaterals Q_A, Q_B, Q_C are adjacent to the outer face, a complete side of each quadrilateral is adjacent to the outer face.

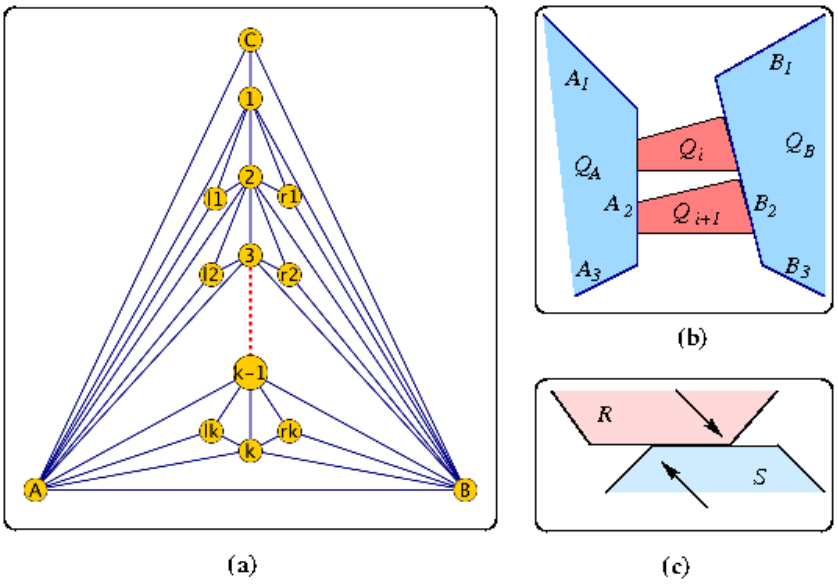

Fig. 2. (a) The graph that provides the counterexample. (b) A pair of subsequent fair quadrilaterals adjacent to the same sides of Q_A and Q_B. (c) Illustration for Lemma 2.

From this observation, we conclude that at most three sides of each of the outer quadrilaterals are inside of the drawing. We consider the three sides A_1, A_2, A_3 and B_1, B_2, B_3 of Q_A and Q_B, respectively, numbered from top to bottom; see Figure 2(b). The quadrilaterals of the chain are adjacent to the three sides in this order, such that if Q_i is adjacent to A_j (resp. B_j), then Q_{i+1} is adjacent to A_k (resp. B_k) with $k \geq j$. The adjacency of each Q_i defines two intervals, one on the polygonal chain A_1, A_2, A_3 and another one on B_1, B_2, B_3.

Observation 2: Consider the $c(= 4)$ corners of Q_A and Q_B, where the sides A_1 and A_2, A_2 and A_3, B_1 and B_2, B_2 and B_3 coincide. Clearly, at most 2 of the intervals that are defined by the adjacencies of the Q_i's are adjacent to each of the c corners. In total, this makes at most $2c = 8$ intervals, that are adjacent to any of the corners of Q_A or Q_B. Hence, at most 8 quadrilaterals of the chain $Q_1, ..., Q_k$ are adjacent to corners of Q_A and/or Q_B.

We now consider the quadrilaterals that do *not* define any of those intervals.

Let Q_i be a quadrilateral that is not adjacent to any of the corners of the polygonal chains A_1, A_2, A_3 and B_1, B_2, B_3. Two of its corners are adjacent to the same side A_k and to the same side $B_l, 1 \leq k, l \leq 3$ of Q_B. We call such a quadrilateral a *fair quadrilateral*.

Lemma 1. *If we choose k large enough, there exists a pair of fair quadrilaterals Q_i and Q_{i+1} that are adjacent to the same sides of Q_A and Q_B.*

Proof: We use a counting argument. We know that at most 8 quadrangles are not fair. Hence, for $k \geq 2 \cdot 2c + 2 = 18$, there must be a pair of subsequent fair quadrilaterals. The worst case happens for $k = 17$ if $Q_2, Q_4, Q_6, \ldots Q_{16}$ are not fair. We can state even more precisely that there are at least $k - 17$ pairs of subsequent fair quadrilaterals. Note that the pair (Q_i, Q_{i+1}) of fair quadrilaterals where Q_i is adjacent to the sides A_1 and B_1, but Q_{i+1} is not adjacent to A_1 and B_1 does not have the property claimed in the lemma. We call such a pair transition pair.

We can partition the set of fair quadrilaterals into at most 5 equivalence classes $C_1, ..., C_5$ that denote the sets of fair quadrilaterals, which are adjacent to the same sides of Q_A and Q_B. When we sweep through the chain of middle quadrilaterals, we simultaneously proceed through the equivalence classes. Hence there exist at most $t = 4$ transition pairs, namely pairs of subsequent fair quadrilaterals that are in different equivalence classes.

These equivalence classes denote the pairs of sides (A_i, B_j) that are used, beginning from the top with, say, (A_1, B_1), then (A_1, B_2), (A_2, B_2), (A_3, B_2) and finally (A_3, B_3). Note that this is not the only possible set of equivalence classes, but by planarity, it is not possible to have (A_2, B_3) and (A_3, B_1) simultaneously. Hence, there are at most 5 classes.

We repeat our counting argument from above and argue that for $k \geq 23$ there are at least 5 or more pairs of subsequent fair quadrilaterals, so at least one has the property claimed in the lemma. □

Before we continue with the proof of the theorem, we include the following Lemma, illustrated in Figure 2(c):

Lemma 2. *If there are two regions R, S touching in some nontrivial interval $I = (a, b)$ then at a, there is a corner of R or S. The same holds for corner b.*

Now, let (Q_i, Q_{i+1}) be a pair of fair same-sided quadrilaterals, touching sides A_p and B_q. Since Q_i and Q_{i+1} have to be adjacent, the two sides next to each other touch. We can use the above Lemma 2 to show that each interval that is shared by two polygons ends at two of the corners of the two polygons. Since there exist the polygonal regions representing r_i and l_i, it is clear that the interval where Q_i and Q_{i+1} touch is disjoint from the regions Q_A and Q_B. Hence the corners derived from Lemma 2 are not the corners of Q_i or Q_{i+1} that are incident to sides A_p and B_q. This is a contradiction, since then both Q_i and Q_{i+1} must have at least 5 corners, or one of them has even 6 corners. □

2.2 Five Sides Are Not Enough

If we allow the regions to be pentagons, we have to sharpen the argument a little more.

Lemma 3. *If we choose k large enough, there exists a triple of fair pentagons P_i, P_{i+1}, P_{i+2} that is adjacent to the same sides of P_A and P_B.*

Proof: We prove this along the same lines as before. Now we have four sides with $c = 6$ inner corners of the pentagons P_A and P_B. As before, we can see that at most 12 pentagons of the inner chain are not fair. Since we aim now for triples and not just for pairs, we get a worst case where every third pentagon is not fair. Hence for $k \geq 3 \cdot 2c + 3$, we get at least $k - 38$ fair subsequent pentagons. Next, we estimate the number of transition triples. The number of equivalence classes of pentagons with sides solely on the same side of P_A and P_B is seven. As we deal with triples, this makes a bound of at most 14 transition triples, since we can differentiate transition points between the first two and the last two pentagons of the triple.

Hence, we have to grow k to $38 + 14 = 52$ to ensure that a triple of fair same-sided pentagons exists. □

Theorem 2. *For k sufficiently large, there does not exist a touching polygon representation for G in which all regions have complexity five or less.*

Proof: We choose k to be at least 52. Now, let (P_i, P_{i+1}, P_{i+2}) be a triple of fair same-sided pentagons, touching sides A_p and B_q. Since P_i and P_{i+1} have to be adjacent, the two sides next to each other touch. We can use Lemma 2 that each interval that is shared by two polygons ends at two of the corners of the two polygons. Since there exist the polygonal regions representing r_i and l_i, it is clear that the interval where Q_i and Q_{i+1} touch is disjoint from the regions P_A and P_B. Hence the corners derived from Lemma 2 are not the corners of P_i or P_{i+1} that are incident to sides A_p and B_q. This is a contradiction, since both P_i and P_{i+1} have at least 5 corners, or one of them has even 6 corners. In the case, that P_i and P_{i+1} have exactly 5 corners, we repeat the same argument for P_{i+1} and P_{i+2}. From the second application, we prove the existence of a second additional corner at P_{i+1} or that P_{i+2} has two additional corners at the side opposite to P_{i+1}. In both cases, we get a contradiction. There exists a region with at least 6 corners. □

Note that six-sided polygons are indeed sufficient to represent the graph in Figure 2(a). In particular, for subsequent fair polygons P_i and P_{i+1}, we can use three segments on the lower side of P_i, while the upper side of P_{i+1} consists of only one segment which completely overlaps the middle of the three segments from the lower side of P_i.

3 Touching 6-Sided Polygon Graph Representation

In this section, we present a linear time algorithm that takes as input a planar graph $G = (V, E)$ and produces a representation of G in which all regions are convex polygons with at most six sides each and slopes 0, 1, and -1. As it is based on Kant's algorithm [13] for drawing graphs on a hexagonal grid drawing, we first review his algorithm.

3.1 Kant's algorithm for hexagonal grid drawing

Let $H = (V, E)$ be a 3-connected, 3-planar graph. Note that the dual $D(H)$ is fully triangulated, as each face in the dual corresponds to exactly one vertex in H. So, for f faces in H, we have f vertices in $D(H)$. We first compute a canonical ordering on the vertices of $D(H)$ as defined by de Fraysseix et al. [6]. Let $v_1, ..., v_f$ be the vertices in $D(H)$ in this canonical order.

Kant's algorithm now constructs a drawing for H such that all edges but one have slope -1, 0, or 1, with the one edge with bends lying on the outer face. The typical structure of those drawings is shown in Figure 3.1.

The algorithm incrementally constructs the drawing by adding the faces of H in reverse order of the canonical order of the corresponding vertices in $D(H)$. We let w_i be the vertices of H. Let face F_i correspond to vertex v_i in $D(H)$. The algorithm starts with a triangular region for the face F_f that corresponds to vertex v_f. The vertex w_x which is adjacent to F_f, F_1 and F_2 is placed at the bottom. Let w_y and w_z be the neighbors of w_x in F_f. These three vertices form the corners of the first face F_f. (w_x, w_z) and (w_x, w_y) are drawn upward with equal lengths and slopes -1 and 1, respectively. All the edges on the path between w_y and w_z along F_f are drawn horizontally between the two vertices. From this first triangle, all other faces are added in reverse canonical order to the upper boundary of the drawing region. If a face is completed by only one vertex w_i, this vertex is placed appropriately above the upper boundary such that it can be connected by two edges with slopes -1 and 1, respectively. If the face is completed by a path, then the two end segments of the path have slopes -1 and 1, while the other edges are horizontal. The construction ends when w_1 is inserted, corresponding to the outer face F_1. Note that there is an edge between w_1 and w_x, which is drawn using some bends. This edge is adjacent to the faces F_1 (the outer face) and F_2.

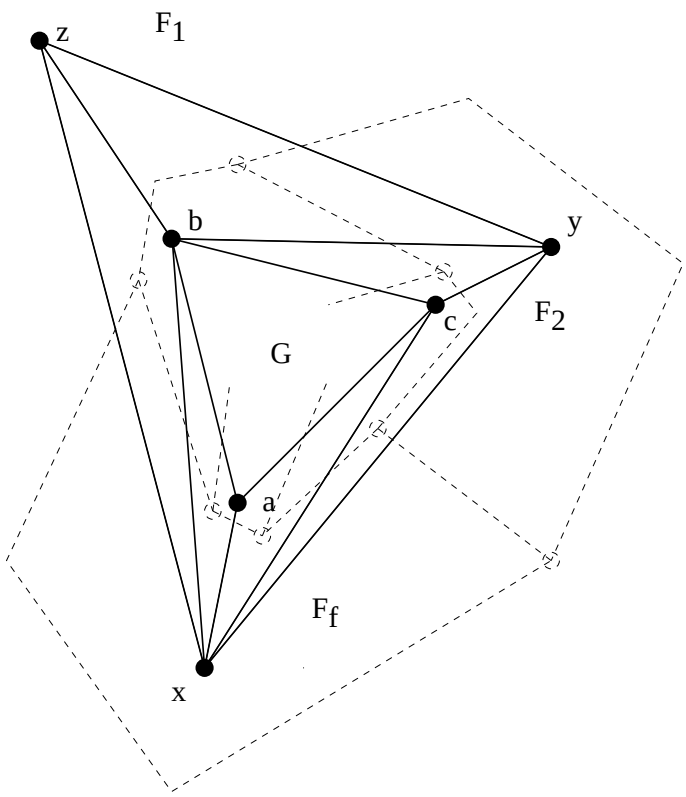

Fig. 3. Polygonal structure obtain from Kant's algorithm.

From this construction, we can observe that the angles at faces $F_f, ..., F_3$ have size $\leq \pi$ as the first two edges do not enter the vertex from above, and the last edge leaves the vertex upwards. Hence, we have the following result.

Lemma 4. *The faces $F_f, ..., F_3$ are convex, and as the slopes of the edges are -1,0 or 1, they are drawn with at most 6 sides.*

This property is exactly what we are aiming for, as the vertices of our input graph G should be represented by convex regions of at most 6 sides. Unfortunately, Kant's algorithm creates two non-convex faces F_1 and F_2 separated by an edge which is not drawn as a line segment. Furthermore, the face F_f is drawn as large as all the remaining faces $F_3, ...F_{f-1}$ together.

Kant also gave an area estimate for the result of his algorithm. A corollary of Kant's algorithm is the following.

Corollary 1. *For a given 3-connected, 3-planar graph H of n vertices, $H - w_x$ can be drawn within an area of $n/2 - 1 \times n/2 - 1$.*

3.2 Application

To apply Kant's result to our problem, we enlarge the embedded input graph G such that the dual of the resulting graph G' can be drawn using Kant's algorithm such that the original vertices of G correspond to the faces $F_3, ..., F_{f-1}$.

We have to add 3 vertices which will correspond to the faces F_1, F_2 and F_f in Kant's algorithm. Since G is fully triangulated, let a, b and c be the vertices at the outer face of G in clockwise order. We add the vertices x, y and z in the outer face and connect to$G$ appropriately. We want z to correspond to the outer face F_1, y correspond to F_2 and x to F_f. First, we add x and connect it to a, b and c such that b and c are still in the outer face. Then we add y and connect it to x, b and c such that b is still in the outer face. Finally, we add z and connect it to x, b and c such that z, y and x are now in outer face, as shown in the Figure 3.2.

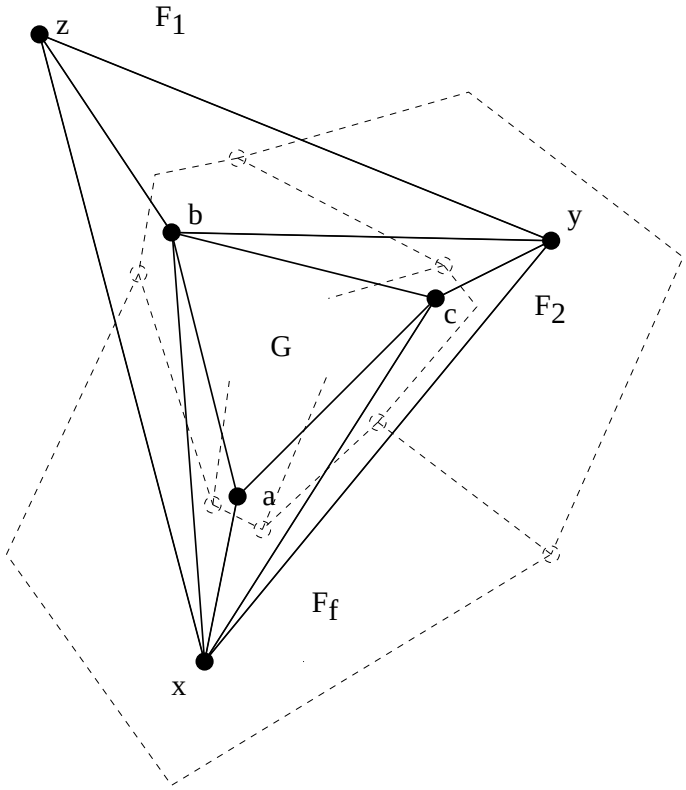

Fig. 4. The graph G enhanced by vertices z, y and x together with its dual which serves as input graph for Kant's algorithm.

Since the vertices x, y, z are on the outer face, we can choose which one is first, second and last in the canonical order. We can then apply Kant's algorithm with the canonical order $v_1 = z, v_2 = y$ and $v_f = x$. After constructing the final drawing, we remove the regions corresponding to vertices z, y and x, leaving us with a hexagonal representation of G. Since

Kant's algorithm runs in linear time, and our emendations can be done in constant time, we can summarize:

Theorem 3. *For a fully triangulated planar graph G on n vertices, we can construct a contact graph of convex hexagons in time $O(n)$. The sides of the hexagons have slope 1, 0, or -1.*

Given any planar graph G, if it is not biconnected, we can make it biconnected using a procedure attributed to Read [22], adding a vertex and two edges at each articulation point. Once biconnected, we can fully triangulate the graph by adding a vertex inside each non-triangular face and connecting that vertex to each vertex on the face. We can then apply Theorem 3, to get a hexagonal representation of the extended graph. Finally, removing the added vertices and their edges, we obtain a hexagonal representation of G. This gives us:

Theorem 4. *For any planar graph G on n vertices, we can construct a contact graph of convex hexagons in time $O(n)$. The sides of the hexagons have slope 1, 0, or -1.*

3.3 Area estimation

For a triangulated input graph $G = (V, E)$, we have n vertices and, by Euler's formula, $2n-4$ faces. Since we enhanced our graph to $n+3$ vertices, we have $f = 2n+2$ faces. Those faces are the vertices in the dual $D(G)$ which is the input to Kant's algorithm. His area estimation gives an area of $n/2 - 1 \times n/2 - 1$ for $f = n$ vertices coalesce the faces F_1, F_2 and F_f into a single outer face by removing the corresponding vertices and edges. Thus, we get an area bound of $n \times n$ using exactly the same argument as he did.

Theorem 5. *For a fully triangulated planar graph G of n vertices, we can achieve a contact representation of convex hexagons with area $n \times n$.*

4 Conclusion and Future Work

We have shown that touching polygonal representations of planar graphs require at least six sides per polygon. We also described a simple, linear time algorithm for generating touching polygon representations for planar graphs that uses at most six sides per polygon, thus matching the lower bound. Moreover the algorithm produces convex polygons defined by sides with fixed slopes of 0, 1, -1, and the entire layout requires $n \times n$ area.

The obvious direction for future work would be a search for characterizations of graphs that can be represented by k-gons where $3 \leq k \leq 5$.

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